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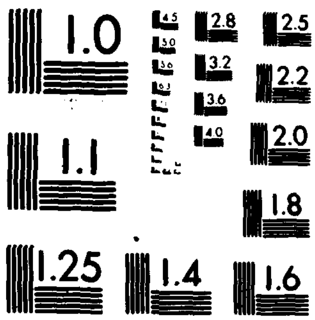

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ANOTHER GENERALIZATION OF CARATHÉODORY'S THEOREM

by

Victor Klee

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13. ABSTRACT

When P is a d-dimensional convex polytope with vertex-set V, ~~we use~~ the term V-simplex is used to denote a d-simplex whose vertices all belong to V. A slight variant of Carathéodory's theorem asserts that for each $v \in V$ there is a collection $\underset{\sim}{S}$ of V-simplices such that $P = \cup\underset{\sim}{S}$ and $v \in \cap\underset{\sim}{S}$. In connection with some constructions in ring theory, Kenneth Goodearl has conjectured there is a collection $\underset{\sim}{S}$ of V-simplices such that $P = \text{con} \cup\underset{\sim}{S}$ and $\dim \cap\underset{\sim}{S} = d$. For $0 \le k < d$ the present note establishes a theorem concerning the generation of P by V-simplices in conjunction with the operation con_{k+1}, where $\text{con}_n X$ is the set of all convex combinations of n or fewer points of X. When $k = 0$ the theorem is Carathéodory's and when $k = d-1$ it is a slight sharpening of Goodearl's conjecture.

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ANOTHER GENERALIZATION OF CARATHÉODORY'S THEOREM

VICTOR KLEE

When P is a d-dimensional convex polytope with vertex-set V, we use the term V-*simplex* to denote a d-simplex whose vertices all belong to V. A slight variant of Carathéodory's theorem [2] asserts that for each $v \in V$ there is a collection $\mathfrak{S}$ of V-simplices such that $P = \cup\mathfrak{S}$ and $v \in \cap\mathfrak{S}$. In connection with some constructions in ring theory, Kenneth Goodearl has conjectured there is a collection $\mathfrak{S}$ of V-simplices such that $P = \operatorname{con} \cup\mathfrak{S}$ and $\dim \cap\mathfrak{S} = d$. (This result is used in [4].) For $0 \le k < d$ the present note establishes a theorem concerning the generation of P by V-simplices in conjunction with the operation con_{k+1}, where $\operatorname{con}_n X$ is the set of all convex combinations of n or fewer points of X. When $k = 0$ the theorem is Carathéodory's and when $k = d-1$ it is a slight sharpening of Goodearl's conjecture.

THEOREM *Suppose that P is a d-dimensional convex polytope with vertex-set V, $0 \le k < d$, and F is a k-face of P. Then there is a collection $\mathfrak{S}$ of V-simplices such that*

$$P = \operatorname{con}_{k+1} \cup\mathfrak{S} \qquad \text{and} \qquad \dim (F\cap(\cap\mathfrak{S})) = k.$$

When $k = d-1$ the intersection $\cap\mathfrak{S}$ is d-dimensional. If V is in general position then con_{k+1} may be replaced by $\operatorname{con}_{\lceil d/(d-k)\rceil}$.

Proof. Observe first that if H is a $(j-1)$-flat in a j-flat G, Q is one of the two closed halfspaces into which H divides G, and $\mathfrak{B}$ is a finite collection of j-dimensio convex subsets of Q such that the set $C = H\cap(\cap\mathfrak{B})$ is $(j-1)$-dimensional, then $\cap\mathfrak{B}$ is j-dimensional. Indeed, choose points c and q in the relative interiors of C and Q respectively, and note that for each $B \in \mathfrak{B}$ there exists $\lambda_B > 0$ such that $(1-\lambda_B)c+\lambda_B q$ With $\varepsilon = \min \{\lambda_B : B \in \mathfrak{B}\} > 0$, $\cap\mathfrak{B}$ contains the j-dimensional set

$$\operatorname{con} (C \cup\{(1-\varepsilon)c+\varepsilon q\}).$$

Whenever P is a d-polytope with vertex-set V, $0 \le k \le d$, and $F_o \subset F_1 \subset \ldots \subset F_k$ is a sequence of faces of P with $\dim F_i = i$ for each i, let $\mathfrak{S}_P(F_o, \ldots, F_k)$ denote the collection of all sets of the form $\mathrm{con}\{v_o, \ldots, v_d\}$ such that

(i) for $0 \le i \le k$, $v_i \in F_i$

(ii) for $1 \le i \le d$, $v_i \in V \sim \mathrm{aff}\,\{v_o, \ldots, v_{i-1}\}$.

Plainly each member of $\mathfrak{S}_P(F_o, \ldots, F_k)$ is a V-simplex. A straightforward induction on i, based on the observation of the preceding paragraph, shows that for $0 \le i \le k$,

$$\dim \cap \mathfrak{S}_{F_i}(F_o, \ldots, F_i) = i.$$

To construct the $\mathfrak{S}$ whose existence is claimed by the theorem, simply set $\mathfrak{S} = \mathfrak{S}_P(F_o, \ldots, F_k)$ for an arbitrary sequence of faces $F_o \subset F_1 \subset \ldots \subset F_k$ with $F_k = F$ and $\dim F_i = i$ for all i. Plainly $\dim (F \cap (\cap S)) = k$, for $\cap\mathfrak{S} \supset \cap\mathfrak{S}_{F_k}(F_o, \ldots, F_k)$. And since

$$\mathfrak{S}_P(F_o, \ldots, F_{d-1}) = \mathfrak{S}_P(F_o, \ldots, F_{d-1}, P),$$

$\cap\mathfrak{S}$ is d-dimensional when $k = d-1$.

It remains to show that $P = \mathrm{con}_r \cup \mathfrak{S}$ with $r = k+1$ in general and $r = \lceil d/(d-k) \rceil$ (the smallest integer $\ge d/(d-k)$) when V is in general position. With $v_o \in F_o$, consider an arbitrary point $p \in P \sim \{v_o\}$ and let q be the last point of the ray from v_o through p that belongs to P. If $q \in \mathrm{con}_r \cup \mathfrak{S}$ then $p \in \mathrm{con}_r \cup \mathfrak{S}$ because $p \in]v_o, q]$ and each member of $\mathfrak{S}$ is a convex set that contains v_o.

Let j denote the dimension of the smallest face G of P that contains q. By Carathéodory's theorem, $q \in \mathrm{con}\, X$ for an affinely independent set X consisting of $j+1$ points of $V \cap G$. If $G \subset F_k$ then $j < k$ and for each $x \in X$ there is a member S_x of $\mathfrak{S}$ which contains x. Hence $q \in \mathrm{con}_k \cup \mathfrak{S}$.

Suppose, on the other hand that $G \not\subset F_k$, and let W be the vertex-set of an arbitrary member of $\mathfrak{S}_{F_k}(F_o, \ldots, F_k)$. Let $m+1$ denote the cardinality of the

maximal affinely independent subsets of $W \cup X$. From the facts that $W \not\subset G$ and $X \not\subset F_k$ it follows that $m > k$ and $m < j$. Since W is affinely independent, there is a set $Y \subset X$ such that the set $W \cup Y$ is affinely independent and of cardinality $m+1$, whence $|Y| = m-k$. Plainly $W \cup Y$ lies in a member of $\mathcal{S}$, as does each of the $(j+1)-(m-k)$ remaining points of X. Hence $p \in \mathrm{con}_{r+1} \cup \mathcal{S}$ with $r = (j+1)-(m-k) \leq k$.

Now suppose, finally, that the vertex-set V of P is in general position, meaning that each set of $d+1$ points of V is affinely independent. Then all proper faces of P are simplices, and $\mathcal{S}$ consists merely of all V-simplices that contain $\bar{F}_k$. Consider v_o, p, q, G, X, and W as described earlier. Then $W \cup Y$ is affinely independent for each set $Y \subset X \sim W$ with $|Y| \leq d-k$. Hence $X \sim W$ can be covered by $\lceil (j+1)/(d-k) \rceil$ members of $\mathcal{S}$, and since $j < d$ it follows that q (and hence p) belongs to $\mathrm{con}_{\lceil d/(d-k) \rceil} \cup \mathcal{S}$. That completes the proof.

To see that the theorem cannot be improved by reducing the subscripts $k+1$ and $\lceil d/(d-k) \rceil$, consider a d-polytope $P = \mathrm{con}\, V$ where V is the union of the vertex-set W of a k-simplex F and the vertex-set X of a $(d-1)$-simplex. Let $\mathcal{S}$ be the collection of all V-simplices S such that $\dim(F \cap S) = k$. Then $|X \cap S| = d-k$ for each $S \in \mathcal{S}$, whence the centroid of $\mathrm{con}\, X$ does not belong to $\mathrm{con}_{\lceil d/(d-k) \rceil - 1} \cup \mathcal{S}$. If a translate W' of W is contained in X then $|W' \cap S| = 1$ for each $S \in \mathcal{S}$, whence the centroid of $\mathrm{con}\, W'$ does not belong to $\mathrm{con}_k \cup \mathcal{S}$.

y other generalizations of Carathéodory's theorem appear in the re. Some of them can be found in the references below.

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